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On the solution of the dynamic Eshelby problem for inclusions of various shapes

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Abstract

In many dynamic applications of theoretical physics, for instance in electrodynamics, elastodynamics, and materials sciences (dynamic variant of Eshelby's inclusion and inhomogeneity problems) the solution of the inhomogeneous Helmholtz equation ('dynamic' or Helmholtz potential) plays a crucial role. In materials sciences from such a solution the dynamical fields due to harmonically transforming eigenfields can be constructed. In contrast to the static Eshelby's inclusion problem (Eshelby, 1957), due to its mathematical complexity, the dynamic variant of the problem is comparably little touched. Only for a restricted set of cases, namely for ellipsoidal, spheroidal and continuous fiber-inclusions, analytical approaches exist. For ellipsoidal shells we derive a 1D integral representation of the Helmholtz potential which is useful to be extended to inhomogeneous ellipsoidal source regions. We determine the dynamic potential and dynamic variant of the Eshelby tensor for arbitrary source densities and distributions by employing a numerical technique based on Gauss quadrature. We study a series of examples of Eshelby problems which are of interest for applications in materials sciences, such as for instance cubic and prismatic inclusions. The method is especially useful to be applied in self-consistent methods (e.g. the effective field method) if one looks for the effective dynamic characteristics of the material containing a random set of inclusions.

Keywords: Dynamic Eshelby inclusion problem; Helmholtz potential; Retarded potential; Ellipsoidal source; Dynamical transforming inclusion; Inhomogeneous inclusions of various shapes; Ellipsoidal inclusion; Elastic wave propagation; Dynamic Green's function

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1. Introduction

The determination of the elastic fields due to dynamically transforming ellipsoidal inclusion is of fundamental interest in a wide range of physics and materials science problems for instance for the non-destructive evaluation (NDE) (Pao, 1978). We derive the dynamic variant of the Eshelby tensor for inhomogeneous time dependent eigenstrain for both the frequency and the time domain. The problem is reduced to the determination of solutions of the inhomogeneous Helmholtz and wave equation in the frequency and time domain, respectively. The availability of 1D-integral representations of the solutions is highly desirable from the essential and practical point of view. We derive such representation for the Helmholtz potential of an ellipsoidal shell. The method is useful to be extended to inhomogeneous ellipsoids to determine the dynamic Eshelby tensor of inclusions with spatially inhomogeneous eigenstrain.

Fu and Mura (1982) have considered integrals, which occur when looking for the dynamic potential of an ellipsoid of inhomogeneous densities. However, they confined in their evaluation to the quasi-static case.

There are only a few cases for which closed form solutions exist, namely for homogeneous spherical inclusions (Mikata and Nemat-Nasser, 1990) and for homogeneous continuous fiber inclusions (Michelitsch et al., 2002, 2003a). For homogeneous ellipsoidal inclusions the solution of the dynamic Eshelby problem was reduced recently to simple surface integrals (Michelitsch et al., 2003a).

The paper is organized as follows: In Section 2 we derive the dynamic Eshelby tensor for inclusions of arbitrary shapes and spatially inhomogeneous eigenstrains which is embedded into an isotropic three-dimensional infinite elastic medium having the same material characteristics (elastic constants and mass density) as the matrix material. The problem is traced back to determine the dynamic potentials ('Helmholtz potentials') in the frequency domain and the retarded potentials (solutions of the inhomogeneous wave equation) in the time domain. In Section 3 we determine the retarded potentials of inclusions of spatially inhomogeneous source densities and arbitrary shapes numerically. Closed form representations are obtained for inhomogeneous spherical inclusions. The method is based on Gauss quadrature. The accuracy and efficiency of the numerical method is confirmed by means of comparisions of the potentials with closed form results of the spherical inclusions. In Secrion 4 we derive an 1D integral representation for the dynamic potential of an homogeneous ellipsoidal shell which is useful to be extended to spatially inhomogeneous ellispoidal inclusions.

2. Dynamic Eshelby inclusion problem

We consider a homogeneous material with elastic constants C_{ijrs}. The constitutive relations are

$$\sigma_{ij} = C_{ijrs}(\varepsilon_{rs} - \varepsilon^*_{rs}) \tag{1}$$

where $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}^*$, $\mathbf{C}$ denote stress, strain, eigenstrain and the tensor of elastic constants, respectively, with

$$\varepsilon_{rs} = \frac{1}{2}(\partial_r u_s + \partial_s u_r) \tag{2}$$

where $\mathbf{u}$ denotes the displacement vector field. The inclusion S is assumed to undergo a *uniform* time dependent transformation with a spatially *non-uniform* eigenstrain $\boldsymbol{\varepsilon}^*$ of the form

$$\boldsymbol{\varepsilon}^*(\mathbf{r}, t) = f(t)\rho_S(\mathbf{r})\boldsymbol{\varepsilon}^0 \tag{3}$$

and with a "density" function $\rho_S(\mathbf{r})$ given by

$$\rho_S(\mathbf{r}) = \Theta_S(\mathbf{r})\chi(\mathbf{r}) \tag{4}$$

where $\mathbf{r} = (x_i) = (x, y, z)$ denotes the space point and $\Theta_S(\mathbf{r})$ denotes the characteristic function of inclusion [1] and $\boldsymbol{\varepsilon}^0$ is a symmetric and *constant* tensor. In (4) we have introduced the *scalar* function $\chi(\mathbf{r})$ which characterizes the spatial variation of the eigenstrain. The function $f(t)$ is an arbitrary function of time with the only restriction $\int_{-\infty}^{\infty} | f(t) | \,\mathrm{d}t < \infty$. Function $f(t)$ describes the transformation evolution of the eigenstrain. We assume the absence of external body forces. Then the equations of motion are given by

$$\rho_\mathrm{m} \frac{\partial^2}{\partial t^2} u_i = \partial_j \sigma_{ij} \tag{5}$$

where ρ_m denotes the *mass density* of the material. With (1) and (2) this equation assumes the form in the Fourier transformed space-frequency representation

$$(\mathbf{T}(\nabla) + \rho_\mathrm{m}\omega^2 \mathbf{1})\mathbf{u}(\mathbf{r}, \omega) + \mathbf{f}^*(\mathbf{r}, \omega) = 0 \tag{6}$$

where $T_{ij}(\nabla) = C_{ikjl}\partial_k\partial_l$, $\mathbf{1}$ denotes the unit tensor and $\mathbf{f}^*(\mathbf{r}, \omega)$ is the effective force density that induces the same displacement field $\mathbf{u}$ in the undisturbed matrix as the inclusion and is obtained as

$$f_i^*(\mathbf{r}, \omega) = -\tilde{f}(\omega) C_{ijrs} \varepsilon_{rs}^0 \partial_j \rho_S(\mathbf{r}) \tag{7}$$

where $\tilde{f}(\omega)$ is the Fourier transform of function $f(t)$ which is defined by

$$\tilde{f}(\omega) = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\omega t} f(t)\mathrm{d}t \tag{8}$$

In the following we assume $f(t) = \delta(t)$ with $\tilde{f}(\omega) = 1$. The displacement field $\mathbf{u}(\mathbf{r}, \omega)$ can be expressed by the dynamic Green's function in the form

$$\mathbf{u}(\mathbf{r}, \omega) = \int \hat{\mathbf{G}}(\mathbf{r} - \mathbf{r}', \omega)\mathbf{f}^*(\mathbf{r}', \omega) d^3\mathbf{r}' \tag{9}$$

The dynamic Green's function $\hat{\mathbf{G}}(\mathbf{r}, \omega)$ in the frequency domain is defined by

$$(\mathbf{T}(\nabla) + \rho_\mathrm{m}\omega^2 \mathbf{1})\hat{\mathbf{G}}(\mathbf{r}, \omega) + \mathbf{1}\delta^3(\mathbf{r}) = 0 \tag{10}$$

where $\delta^3(\mathbf{r})$ denotes the spatial δ-function. To determine the displacement field $\mathbf{u}$ it is convenient to introduce a tensor function $\mathbf{G}(\mathbf{r}, \omega)$ which is defined by

$$(\mathbf{T}(\nabla) + \rho_\mathrm{m}\omega^2 \mathbf{1})\mathbf{G}(\mathbf{r}, \omega) + \mathbf{1}\rho_S(\mathbf{r}) = 0 \tag{11}$$

Obviously $\mathbf{G}$ and $\hat{\mathbf{G}}$ are then related by

$$\mathbf{G}(\mathbf{r}, \xi, \omega) = \int \hat{\mathbf{G}}(\mathbf{r} - \mathbf{r}', \omega)\rho_S(\mathbf{r}')\mathrm{d}^3\mathbf{r}' = \int_S \hat{\mathbf{G}}(\mathbf{r} - \mathbf{r}', \omega)\chi(\mathbf{r}')\mathrm{d}^3\mathbf{r}' \tag{12}$$

where ξ indicates a set of geometric characteristics of the inclusion (for instance the semi-axes a_i in the case of an ellipsoidal inclusion). For isotropic material and inclusions of arbitrary shapes the dynamic potential function (12) has the form (Michelitsch et al., 2002, 2003a)

$$\mathbf{G}(\mathbf{r}, \xi, \omega) = \frac{1}{\rho_\mathrm{m}\omega^2}[\beta_2^2 g(\mathbf{r}, \xi, \beta_2)\mathbf{1} - \nabla \otimes \nabla\{g(\mathbf{r}, \xi, \beta_1) - g(\mathbf{r}, \xi, \beta_2)\}] \tag{13}$$

with the two frequencies β_i given by

$$\beta_1 = \frac{\omega}{c_1} = \omega\sqrt{\frac{\rho_\mathrm{m}}{\lambda + 2\mu}}, \quad \beta_2 = \frac{\omega}{c_2} = \omega\sqrt{\frac{\rho_\mathrm{m}}{\mu}} \tag{14}$$

[1] $\Theta_S(\mathbf{r}) = 1, \mathbf{r} \in S$ and $\Theta_S(\mathbf{r}) = 0, \mathbf{r} \notin S$

corresponding to one longitudinal and two transversal acoustic waves of the isotropic 3D medium where λ and μ denote the Lame constants. $g(\mathbf{r}, \xi, \beta)$ in (13) is defined by the inhomogeneous Helmholtz equation

$$(\Delta + \beta^2)g(\mathbf{r}, \xi, \beta) + \rho_S(\mathbf{r}) = 0 \tag{15}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. We call function g the *dynamic potential* of the source density ρ_S. The time domain representation of (13) yields (Michelitsch et al., 2002, 2003a) [2]

$$\mathbf{G}(\mathbf{r}, \xi, t) = \frac{1}{\mu} g_2(\mathbf{r}, \xi, t)\mathbf{1} + \frac{1}{\rho_{\mathrm{m}}} \nabla \otimes \nabla \{h_1(\mathbf{r}, \xi, t) - h_2(\mathbf{r}, \xi, t)\} \tag{16}$$

where the functions $g_i(\mathbf{r}, \xi, t)$ and $h_i(\mathbf{r}, \xi, t)$ are determined by

$$\left(\Delta - \frac{1}{c_i^2}\frac{\partial^2}{\partial t^2}\right) g_i(\mathbf{r}, \xi, t) + \delta(t)\rho_S(\mathbf{r}) = 0 \tag{17}$$

and

$$\frac{\partial^2}{\partial t^2} h_i(\mathbf{r}, \xi, t) = g_i(\mathbf{r}, \xi, t) \tag{18}$$

With Eqs. (13) we can derive the dynamic Eshelby tensor for both the frequency and the time domain in terms of the dynamic potentials g_i. With Eqs. (7), (9), and (12) we can write for the displacement field

$$u_l(\mathbf{r}, \xi) = -C_{kjrs}\varepsilon^0_{rs}\partial_j G_{kl}(\mathbf{r}, \xi) \tag{19}$$

where $\mathbf{G}$ corresponds to Eqs. (13) and (16) in the frequency and time domain, respectively. The strain can be written as

$$\varepsilon_{il}(\mathbf{r}, \xi) = -C_{kjrs}\varepsilon^0_{rs}(P_{ijkl}(\mathbf{r}, \xi))_{(il)} \tag{20}$$

where (il) indicates symmetrization with respect to the subscripts il. Moreover it is denoted

$$P_{ijkl}(\mathbf{r}, \xi) = \partial_i\partial_j G_{kl}(\mathbf{r}, \xi) \tag{21}$$

Using (20) we define the *dynamic Eshelby tensor S* analogously to statics by

$$\varepsilon_{il}(\mathbf{r}, \xi) = S_{ilrs}(\mathbf{r}, \xi)\varepsilon^0_{rs} \tag{22}$$

where the *dynamic Eshelby tensor S* is given by

$$S_{ilrs}(\mathbf{r}, \xi) = -C_{kjrs}(P_{ijkl}(\mathbf{r}, \xi))_{(il)} \tag{23}$$

This relation covers both, the time and frequency domain. It holds for arbitrary density functions $\rho_S(\mathbf{r})$. Hence inclusions of *arbitrary shapes* and density functions $\chi(\mathbf{r})$ are covered. P_{ijkl} is a spatially *non-uniform* tensor function inside an ellipsoidal inclusion with homogeneous eigenstrain. When we expand P_{ijkl} in the frequency domain into a series with respect to frequency ω, the zero-order in ω corresponds to the *static* Eshelby tensor. For homogeneous eigenstrain (corresponding to $\chi = 1$) Eshelby's classical result of 1957 is reproduced (Michelitsch et al., 2003a). Derivations of the dynamic potential g defined by Eqs. (15) and (17) in the frequency and time domain, respectively, can be found for the 3D space (Michelitsch et al., 2003a), as well as for the 2D space (Wang et al., 2003). In that paper a numerical method to generate dynamic potentials for inhomogeneous source regions of arbitrary shapes was proposed recently. Here we also follow this approach to generate the subsequent examples (Figs. 1–4).

[2] $\mathbf{G}(\mathbf{r}, \xi, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega t}\mathbf{G}(\mathbf{r}, \xi, \omega)\mathrm{d}\omega$.

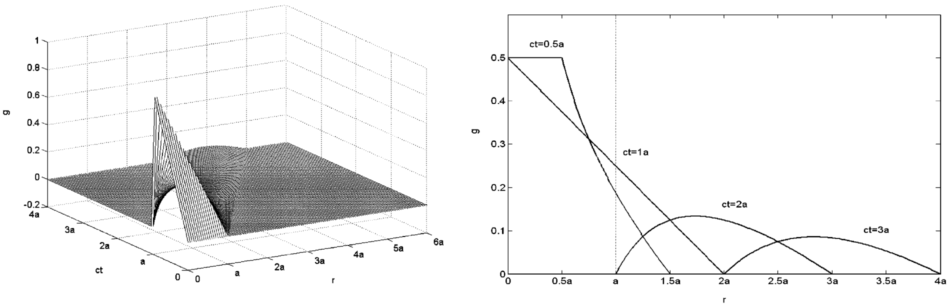

Fig. 1. Time evolution of the retarded potential of a solid spherical source with radius a (Eq. (26)) for different times t.

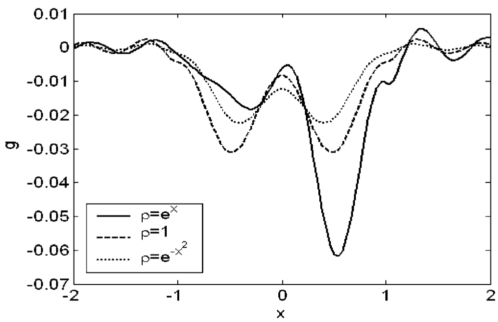

Fig. 2. Dynamic potentials for ellipsoidal sources for different densities $\rho_S = \Theta(1-P)\chi(x)$ ($\chi(x) = 1, x, x^2$, respectively) vs. $\mathbf{r} = (x,0,0)$, ($a_1 = 1, a_2 = 0.7, a_3 = 0.4, \beta = 10$).

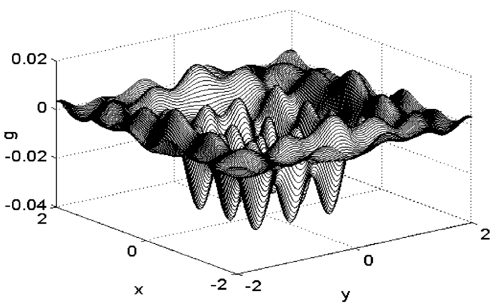

Fig. 3. Dynamic potential of the cubic inclusion (cross-section $z = 0$, and $\beta = 10$).

3. Retarded potentials of inhomogeneous inclusions

In this section we determine *numerically* the dynamic potential of (15) and its time domain representation, the retarded potential defined in (17) by evaluating the convolution

$$g(\mathbf{r}) = \int \hat{g}(\mathbf{r} - \mathbf{r}')\rho_S(\mathbf{r}')d^3\mathbf{r}' \tag{24}$$

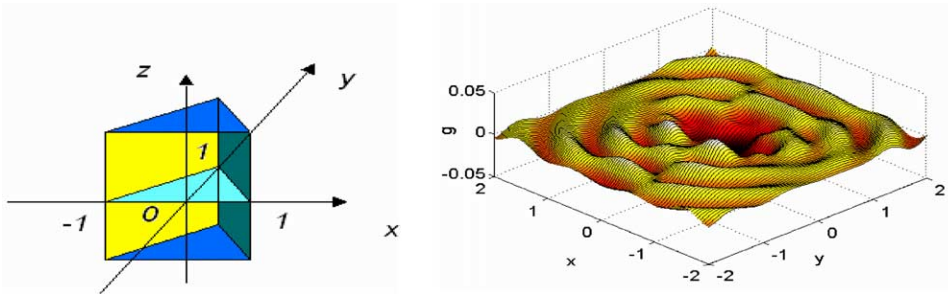

Fig. 4. Cross-section of the dynamic potential of a prismatic source of unit density vs. $\mathbf{r} = (x, 0, y)(\beta = 10)$.

In the frequency domain $\hat{g}(R, \beta) = \frac{e^{i\beta R}}{4\pi R}$ denotes the *Helmholtz Green's function* [3] and in the time domain $\hat{g}(R, t) = \frac{\delta\left(t - \frac{R}{c}\right)}{4\pi R}$ denotes the *retarded Green's function.* [4] For the numerical evaluation of the δ-function in the time domain Green's function we make use of

$$\frac{\delta\left(t - \frac{R}{c}\right)}{4\pi R} = \lim_{\varepsilon \to 0+} \frac{e^{-(t-R/c)^2/(4\varepsilon)}}{8\pi R\sqrt{\pi\varepsilon}} \tag{25}$$

To compute integral (24) we use the Gauss–Chebyshev quadrature formula (Press et al., 1992) In Fig. 1 the retarded potential of a homogeneous solid sphere is drawn according to a source density $\delta(t)\Theta(a - r)$ ($a =$ inclusion radius) which we obtain in closed form

$$\begin{aligned} g(r, a, t) &= g^{\text{in}}\Theta(a - r) + g^{\text{out}}\Theta(r - a) \\ g^{\text{in}}(r, a, t) &= c^2 t\Theta(a - r - ct) + \frac{c}{4r}(a^2 - (ct - r)^2)\Theta(r^2 - (ct - a)^2) \\ g^{\text{out}}(r, a, t) &= \frac{c}{4r}(a^2 - (ct - r)^2)\Theta(a^2 - (ct - r)^2) \end{aligned} \tag{26}$$

where g^{in} and g^{out} denote the potentials of the internal space ($r < a$) and the external space ($r > a$), respectively. Fig. 1 shows that the excitation $\delta(t)\Theta(a - r)$ generates an outgoing spherical wave package of wavelength $2a$ which remains constant due to the absence of dispersion. Due to the finite propagation velocity c the wave package arrives at an external point $\mathbf{r}$ at $(r - a)/c$ which is the runtime form the closest source point located on the boundary of the sphere. Moreover a remarkable superposition effect takes place in the *internal space* at space-point $\mathbf{r}$ for $t < (a - r)/c$: In this time range the potential increases linearly in time (for $0 < t < (a - r)/c$ is $g^{\text{in}} = c^2 t$). For a detailed discussion we refer the reader to our recent paper (Wang et al., 2003). In Fig. 2 the real part g (frequency domain) for an *inhomogeneous* ellipsoidal inclusion of density ($\rho_S = \Theta(1 - P)\chi$, $\chi = 1, x, x^2$, respectively, $P^2 = \frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} + \frac{z^2}{a_3^2}$) is drawn. The symmetry $x \leftrightarrow -x$ of $\chi = 1, x^2$ is reflected by the potential. Moreover the effect of breaking this symmetry is shown for $\chi = x$. In the frequency domain spatial oscillations occur due to a finite frequency ($\beta = \frac{\omega}{c}$).

Fig. 3 shows a 3D plot (vs. plane $z = 0$) of the real part of g for a *homogeneous cubic inclusion* located in the region $|x_i| < 1$ ($\rho_S = \Theta(1 - x^2)\Theta(1 - y^2)\Theta(1 - z^2)$). The cubic symmetry of the inclusion is reflected by g. The frequency domain representation (real part) of g for a *homogeneous triangular prismatic* inclusion ($\chi = 1$) is drawn in Fig. 4. The 3D plot show the potential in the plane $z = 0$ (Fig. 4), respectively. The prismatic inclusion covers the region $|x| < 1$, $0 \leqslant y \leqslant 1 - |x|$, $|z| < 1$. Fig. 4 indicates that the symmetry of the prism $x \leftrightarrow -x$ and $z \leftrightarrow -z$. The prism does not possess symmetry $y \leftrightarrow -y$ and is therefore also absent.

[3] $\hat{g}(R, \beta)$ is defined by $(\Delta + \beta^2)\hat{g}(r, \beta) + \delta^3(\mathbf{r}) = 0$.

[4] $\hat{g}(R, t)$ is defined by $(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\hat{g}(r, t) + \delta^3(\mathbf{r})\delta(t) = 0$.

4. Dynamic potential of a homogeneous ellipsoidal shell

Here we developed a 1D integral representation of the Helmholtz potential for source densities represented by a infinitely thin homogeneous ellipsoidal shell. The approach is useful to be extended to inhomogeneous source regions of ellipsoidal shape to determine the dynamic Eshelby tensor in terms of 1D integrals. For the Newtonian potentials of statics of inhomogeneous ellipsoidal source regions numerous papers exist. Key contributions were given e.g. by Ferrers (1877), Dyson (1891), Levin and Muratov (1971), and Rahman (2001) among many others referenced therein.

Let us consider an ellipsoidal source region S with the semi-axes a_i. The quantity

$$P^2 = \frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} + \frac{z^2}{a_3^2} \tag{27}$$

characterizes the internal and external space with $P < 1$ and $P > 1$. First we consider the *dynamic potential* of a *solid ellipsoid* which is determined by the inhomogeneous Helmholtz equation

$$(\Delta + \beta^2)g + \rho = 0 \tag{28}$$

ρ represents the source density. For a density being a function $f(P^2)$ within the ellipsoid, the dynamic potential g is defined by

$$(\Delta + \beta^2)g + \Theta(1 - P)f(P^2) = 0 \tag{29}$$

Here we introduced the Heaviside step function $\Theta(\xi)$ which is defined by $\Theta(\xi) = 1$ if $\xi > 0$ and $\Theta(\xi) = 0$ if $\xi < 0$. g is determined by the Fourier integral

$$g = \frac{1}{(2\pi)^3} \int e^{i\mathbf{k}\cdot\mathbf{r}} \frac{\tilde{\rho}(\mathbf{k})}{k^2 - \beta^2} d^3k \tag{30}$$

where $\tilde{\rho}$ is the Fourier transform of the density ρ defined by

$$\tilde{\rho}(\mathbf{k}) = \int e^{-i\mathbf{k}\cdot\mathbf{r}} \rho(\mathbf{r}) d^3r \tag{31}$$

For the the spatial coordinates we put $x_i = Pa_in_i$ (where $\mathbf{n}^2 = 1$). Then g can be rewritten as

$$g(\mathbf{r}, \mathbf{a}, \beta) = \int_0^1 dp' f(p'^2) p' \Phi(\mathbf{r}, p', \beta) \tag{32}$$

It was shown recently (Michelitsch et al., 2003b) that the present approach for the dynamic potential of an inhomogeneous density of the form $f(P^2)$ reproduces in the static limit $\beta \rightarrow 0$ classical results of Ferrers (1877) and Dyson (1891). In view of (32) we observe that

$$(\Delta + \beta^2)\Phi(\mathbf{r}, p_0, \beta)p_0 + \delta(P - p_0) = 0 \tag{33}$$

$p_0\Phi$ corresponds to the *dynamic potential of an infinitely thin homogeneous ellipsoidal shell* at $P = p_0$ on which we focus in the following considerations ($\beta = \frac{\omega + i\gamma}{c}$ where $\gamma > 0$). The space–time representation yields (Michelitsch et al., 2003b)

$$\Phi(\mathbf{r}, \mathbf{a}, t, p') = \frac{\Theta(t)e^{-\gamma t}}{4\pi} \int_{|\hat{\mathbf{K}}|=1} d\Omega(\hat{\mathbf{K}}) s^2(\hat{\mathbf{K}}) \left[\delta\left(t - \frac{s}{c}(P\hat{\mathbf{K}} \cdot \mathbf{n} + p')\right) - \delta\left(t + \frac{s}{c}(P\hat{\mathbf{K}} \cdot \mathbf{n} + p')\right)\right] \tag{34}$$

where $s(\hat{\mathbf{K}}) = 1 \Big/ \sqrt{\left(\frac{\hat{K}_1^2}{a_1^2} + \frac{\hat{K}_2^2}{a_2^2} + \frac{\hat{K}_3^2}{a_3^2}\right)}$. (34) represents the *causal* space-time representation of the dynamic potential, the *retarded potential* of the *shell* of an ellipsoid with semi-axes $p'a_i$ and fulfills the wave equation

$$\left(\Delta - \frac{1}{c^2}\left[\frac{\partial}{\partial t} + \gamma\right]^2\right)(p'\Phi(\mathbf{r}, \mathbf{a}, t, p')) + \delta(t)\delta(P - p') = 0 \tag{35}$$

The frequency-space domain representation of (34) is defined by the Fourier transform

$$\Phi(\mathbf{r}, \mathbf{a}, \beta, p') = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t}\Phi(\mathbf{r}, \mathbf{a}, t, p')\mathrm{d}t \tag{36}$$

and yields (Michelitsch et al., 2003b) ($x_i = Pa_i\, n_i$)

$$\Phi(Pa_in_i, \beta, p') = \frac{1}{4\pi}\int_{|\hat{\mathbf{K}}|=1} \mathrm{d}\Omega(\hat{\mathbf{K}})s^2(\hat{\mathbf{K}})[\Theta(P\hat{\mathbf{K}}\cdot\mathbf{n} + p')\mathrm{e}^{\mathrm{i}\beta s(\hat{\mathbf{K}})[(P\hat{\mathbf{K}}\cdot\mathbf{n}+p')]} \quad - \Theta(-(P\hat{\mathbf{K}}\cdot\mathbf{n} + p'))\mathrm{e}^{-\mathrm{i}\beta s(\hat{\mathbf{K}})[(P\hat{\mathbf{K}}\cdot\mathbf{n}+p')]}] \tag{37}$$

This equation covers both the *external* ($P > 1$) and *internal* space ($P < 1$). For the *internal space* it assumes the form

$$\Phi(Pa_in_i, \beta, p') = \frac{1}{4\pi}\int_{|\hat{\mathbf{K}}|=1} \mathrm{d}\Omega(\hat{\mathbf{K}})s^2(\hat{\mathbf{K}})\mathrm{e}^{\mathrm{i}\beta s(\hat{\mathbf{K}})[(P\hat{\mathbf{K}}\cdot\mathbf{n}+p')]} \tag{38}$$

Above relations (37) and (38) can be rewritten ($p' = 1$)

$$\Phi(\mathbf{r}, \beta) = \frac{a_1a_2a_3}{4\pi}\int_{|\hat{\mathbf{n}}'|=1} \frac{\mathrm{d}\Omega(\mathbf{n}')}{r'}[\Theta(\mathbf{r}\cdot\mathbf{n}' + r')\mathrm{e}^{\mathrm{i}\beta(\mathbf{r}\cdot\mathbf{n}'+r')} - \Theta(-(\mathbf{r}\cdot\mathbf{n}' + r'))\mathrm{e}^{-\mathrm{i}\beta(\mathbf{r}\cdot\mathbf{n}'+r')}] \tag{39}$$

which yields for the internal space

$$\Phi(\mathbf{r}, \beta) = \frac{a_1a_2a_3}{4\pi}\int_{|\hat{\mathbf{n}}'|=1} \frac{\mathrm{d}\Omega(\mathbf{n}')}{r'}\mathrm{e}^{\mathrm{i}\beta(\mathbf{r}\cdot\mathbf{n}'+r')} \tag{40}$$

$r'(\mathbf{n}') = \sqrt{a_1^2n_1'^2 + a_2^2n_2'^2 + a_3^2n_3'^2}$ is a parameterization of the radius on the shell. Note that (39), (40) are equivalent representations to the convolution

$$\Phi = \frac{a_1a_2a_3}{4\pi}\int_{|\hat{\mathbf{n}}'|=1} \frac{\mathrm{d}\Omega(\mathbf{n}')}{|\mathbf{r} - \mathbf{r}'|}\mathrm{e}^{\mathrm{i}\beta|\mathbf{r}-\mathbf{r}'|} \tag{41}$$

Let us consider now the real part of the internal potential $\mathrm{Re}\Phi$ of (40) which is given by

$$\mathrm{Re}\Phi(\mathbf{r}, \beta) = \frac{1}{4\pi}\int_{|\hat{\mathbf{K}}|=1} \mathrm{d}\Omega(\hat{\mathbf{K}})s^2(\hat{\mathbf{K}})\cos\left(\beta s(\hat{\mathbf{K}})\left[1 + \sum_i \frac{\hat{K}_ix_i}{a_i}\right]\right) \tag{42}$$

Expanding (42) into a β-power series yields

$$\mathrm{Re}\Phi = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!}\Phi_{2m}\beta^{2m} \tag{43}$$

where the coefficient Φ_{2m} of the power β^{2m} becomes

$$\Phi_{2m} = \frac{1}{4\pi}\int_{|\hat{\mathbf{K}}|=1} \frac{\mathrm{d}\Omega(\hat{\mathbf{K}})}{\left(\frac{\hat{K}_1^2}{a_1^2} + \frac{\hat{K}_2^2}{a_2^2} + \frac{\hat{K}_3^2}{a_3^2}\right)^{m+1}}\left[1 + \sum_i \frac{\hat{K}_ix_i}{a_i}\right]^{2m} \tag{44}$$

Especially for $m = 0$ we identify (44) with the *internal Newtonian potential* of a homogeneous ellipsoidal shell (see e.g. Routh, 1892). Introducing the operator

$$\Delta_{\mathbf{q}} = \sum_i a_i^2 \frac{\partial^2}{\partial x_i^2} \tag{45}$$

where $q_i = \frac{x_i}{a_i}$, we observe that the real part (42) can be generated by the series

$$\mathrm{Re}\Phi(\mathbf{r},\beta) = \frac{1}{4\pi}\sum_{j=0}^{\infty}\frac{\Delta_{\mathbf{q}}^j}{(2j)!}\int_{|\hat{\mathbf{K}}|=1} \mathrm{d}\Omega(\hat{\mathbf{K}}) s^2(\hat{\mathbf{K}}) \cos\left(\beta s(\hat{\mathbf{K}})\sum_i \frac{\hat{K}_i x_i}{a_i}\right) \tag{46}$$

From (41) follows that (44) is a representation of the convolution of the $(2m-1)$th power ($m = 0,1,2,\ldots$) of the distance over the ellipsoid shell, namely

$$\Phi_{2m} = \frac{a_1 a_2 a_3}{4\pi}\int_{|\hat{\mathbf{n}}'|=1} \mathrm{d}\Omega(\mathbf{n}') \mid \mathbf{r}-\mathbf{r}'|^{2m-1} \tag{47}$$

Hence we can write for (44)

$$\Phi_{2m} = \frac{1}{4\pi}\sum_{j=0}^{\infty}\frac{\Delta_{\mathbf{q}}^j}{(2j)!}\int_{|\hat{\mathbf{K}}|=1} \frac{\mathrm{d}\Omega(\hat{\mathbf{K}})}{\left(\frac{\hat{K}_1^2}{a_1^2}+\frac{\hat{K}_2^2}{a_2^2}+\frac{\hat{K}_3^2}{a_3^2}\right)^{m+1}}\left[\frac{\hat{K}_1 x}{a_1}+\frac{\hat{K}_2 y}{a_2}+\frac{\hat{K}_3 z}{a_3}\right]^{2m} \tag{48}$$

Here we have used that all integrals

$$\int_{|\hat{\mathbf{K}}|=1} \mathrm{d}\Omega(\hat{\mathbf{K}})\hat{K}_1^p \hat{K}_2^q \hat{K}_3^r s^{2n}(\hat{\mathbf{K}}) = 0 \tag{49}$$

if at least one of the numbers p,q,r is odd. Hence all contributions of *odd* powers of $\left[\frac{\hat{K}_1 x}{a_1}+\frac{\hat{K}_2 y}{a_2}+\frac{\hat{K}_3 z}{a_3}\right]$ in (44) are vanishing. Therefore (48) which collects only even powers is valid. Hence we only have to consider integrals of the form ($q_i = \frac{x_i}{a_i}$)

$$I_m = \int_{|\hat{\mathbf{K}}|=1} \frac{\mathrm{d}\Omega(\hat{\mathbf{K}})}{\left(\frac{\hat{K}_1^2}{a_1^2}+\frac{\hat{K}_2^2}{a_2^2}+\frac{\hat{K}_3^2}{a_3^2}\right)^{m+1}}[\hat{K}_1 q_1 + \hat{K}_2 q_2 + \hat{K}_3 q_3]^{2m} \tag{50}$$

(24) can be identified with the $2m$th-order of the Taylor expansion of (47), namely

$$I_m = a_1 a_2 a_3 \int \mathrm{d}\Omega(\mathbf{n}')\frac{(\mathbf{r}\cdot\nabla')^{2m}}{(2m)!} r'^{2m-1} \tag{51}$$

This is explicitly confirmed (Michelitsch et al., 2003b) by direct evaluation of (25). Especially it is shown there that I_m can also be represented by

$$I_m = \frac{(2m)!}{2^{2m} m! m!}\int_{|\hat{\mathbf{K}}|=1} \mathrm{d}\Omega(\hat{\mathbf{K}}) s^2(\hat{\mathbf{K}})[r^2 - (\mathbf{r}\cdot\hat{\mathbf{K}})^2]^m \tag{52}$$

and assumes the form

$$I_m = 2\pi\frac{(2m)!}{2^{2m} m! m!}\int_0^{\infty}\frac{u^m \mathrm{d}u}{\sqrt{\left(1+\frac{u}{a_1^2}\right)\left(1+\frac{u}{a_2^2}\right)\left(1+\frac{u}{a_3^2}\right)}}\left(\frac{q_1^2}{1+\frac{u}{a_1^2}}+\frac{q_2^2}{1+\frac{u}{a_2^2}}+\frac{q_3^2}{1+\frac{u}{a_3^2}}\right)^m \tag{53}$$

This representation is useful to generate Φ_{2m} in (48). With (43) we can evaluate (42), taking into account that

$$\sum_{m=0}^{\infty}\frac{(-1)^m}{2^{2m} m! m!}(\sqrt{uT})^{2m}\beta^{2m} = J_0(\beta\sqrt{uT}) \tag{54}$$

where J_0 denotes the zero-order Bassel function of the first kind and $T = (\frac{x^2}{u+a_1^2} + \frac{y^2}{u+a_2^2} + \frac{z^2}{u+a_3^2})$. Hence we obtain for the real part (42) of the internal Helmholtz potential

$$\mathrm{Re}\Phi = \frac{a_1 a_2 a_3}{2} \sum_{j=0}^{\infty} \frac{\Delta_{\mathbf{q}}^j}{(2j)!} \int_0^{\infty} \frac{J_0(\beta\sqrt{uT})\mathrm{d}u}{\sqrt{(u+a_1^2)(u+a_2^2)(u+a_3^2)}} \tag{55}$$

In the static limit $\beta \to 0$ we identify (55) with the *internal Newtonian potential* of an ellipsoidal shell (Routh, 1892). Next we consider the imaginary part ($p' = 1$) of (38), namely

$$\mathrm{Im}\Phi(\mathbf{r}, \beta) = \frac{1}{4\pi} \int_{|\hat{\mathbf{K}}|=1} \mathrm{d}\Omega(\hat{\mathbf{K}}) s^2(\hat{\mathbf{K}}) \sin(\beta s(\hat{\mathbf{K}})(P\hat{\mathbf{K}} \cdot \mathbf{n} + 1)) \tag{56}$$

This expression holds for both the internal and external space. Expanding it with respect to β yields

$$\mathrm{Im}\Phi = \sum_{m=0}^{\infty} \frac{(-1)^m \Phi_{2m+1}}{(2m+1)!} \beta^{2m+1} \tag{57}$$

with

$$\Phi_{2m+1} = \frac{a_1 a_2 a_3}{4\pi} \int_{|\hat{\mathbf{n}}'|=1} \mathrm{d}\Omega(\mathbf{n}') \mid \mathbf{r} - \mathbf{r}' |^{2m} \tag{58}$$

In view of (56) we can write

$$\mathrm{Im}\Phi(\mathbf{r}, \beta) = \frac{\beta}{4\pi} \sum_{j=0}^{\infty} \frac{\Delta_{\mathbf{q}}^j}{(2j+1)!} \int_{|\hat{\mathbf{K}}|=1} \mathrm{d}\Omega(\hat{\mathbf{K}}) s^3(\hat{\mathbf{K}}) \cos\left(\beta s(\hat{\mathbf{K}}) \sum_i \frac{\hat{K}_i x_i}{a_i}\right) \tag{59}$$

which can be rewritten as (Michelitsch et al., 2003b)

$$\mathrm{Im}\Phi(\mathbf{r}, \beta) = a_1 a_2 a_3 \sum_{j=0}^{\infty} \frac{\Delta_{\mathbf{q}}^j}{(2j+1)!} \frac{\sin \beta r}{r} = 4\pi a_1 a_2 a_3 \mathrm{Im} \sum_{j=0}^{\infty} \frac{\Delta_{\mathbf{q}}^j}{(2j+1)!} \hat{g}(r, \beta) \tag{60}$$

where $\hat{g}(r, \beta) = \frac{e^{i\beta r}}{4\pi r}$ is the Helmholtz Green's function and $4\pi a_1 a_2 a_3$ is the "mass" of the ellipsoidal shell. The imaginary part (60) is a homogeneous solution of the Helmholtz equation. Comparing the powers of β in (57) and (60) yields for the integral (58)

$$\Phi_{2m+1} = a_1 a_2 a_3 \sum_{j=0}^{\infty} \frac{\Delta_{\mathbf{q}}^j}{(2j+1)!} r^{2m} \tag{61}$$

With (60) and Φ_{2m+1} we have determined the series (57) which is convergent in the entire space.

5. Conclusion

The dynamic variant of the Eshelby tensor of a three-dimensional infinite isotropic medium with a spatially inhomogeneous, dynamically transforming inclusion with eigenstrain of the type (3) was derived (Eq. (23)), valid in the frequency- and time domain. The problem to determine the dynamic Eshelby tensor was reduced to determine the dynamic potentials defined by Eqs. (15) and (17) for the frequency and time domain, respectively. The time domain representation of the dynamic potentials corresponds to the *retarded* potentials of a given source distribution. Examples of inclusions of different shapes and densities were considered such as spheres, ellipsoids, cubes and triangular prisms.

Furthermore we derived a compact 1D-integral integral representation for the real part of the *internal dynamic potential* of a homogeneous ellipsoidal shell (Eq. (55)). The imaginary part was reduced to a differential operator series (Eq. (60)) holding in the entire space. The availability of such a compact representation for the dynamic potential of a homogeneous ellipsoidal shell is useful for the solution of many problems in mathematical physics. The method is being extended to ellipsoidal source regions with inhomogeneous densities.

The dynamic Eshelby tensor is a main cornerstone for the solution of wide range of *dynamical problems* in the mechanics of materials. Moreover the determination of dynamic potentials in the frequency and time domain is a crucial task in many problems of theoretical physics.

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References

Dyson, F.D., 1891. The potentials of ellipsoids of variable densities. Quart. J. Pure Appl. Math. 30, 259–288.

Eshelby, J.D., 1957. The determination of the elastic field of an ellipsoidal inclusion, and related problems. Proc. R. Soc. London A, 376–396.

Ferrers, N.M., 1877. On the potentials of ellipsoids, ellipsoidal shells, elliptic laminae and elliptic rings of variable densities. Quart. J. Pure Appl. Math., 1–22.

Fu, L.S., Mura, T., 1982. Volume integrals associated with the inhomogeneous Helmholtz equation. Wave Motion, 141–149.

Levin, M.L., Muratov, R.Z., 1971. On the internal potential of heterogeneous ellipsoids. Astrophys. J., 441–445.

Mikata, Y., Nemat-Nasser, S., 1990. Elastic field due to a dynamically transforming spherical inclusion. J. Appl. Mech. ASME 57 (4), 845–849.

Michelitsch, T.M., Levin, V.M., Gao, H., 2002. Dynamic potentials and Green's functions of a quasi-plane piezoelectric medium with inclusion. Proc. R. Soc. Lond. A 458, 2393–2415.

Michelitsch, T.M., Gao, H., Levin, V.M., 2003a. Dynamic Eshelby tensor and potentials for ellipsoidal inclusions. R. Soc. London A. 459, 863–890.

Michelitsch, T.M., Gao, H., Levin, V.M., 2003b. On the dynamic potentials of ellipsoidal shells. Quart. J. Mech. Appl. Math. 56 (4), 629–648.

Pao, Y.H., (Ed.), 1978. Elastic Waves and non-Destructive Testing of materials, AMD-Vol 29, ASME.

Press, W.H., Teukolsky, S.A., Vetterling, W.T., Flannery, B.P., 1992. Numerical Recipes in Fortran 77: the Art of Scientific Computing, second ed. Cambridge University Press.

Rahman, M., 2001. On the newtonian potentials of heterogeneous ellipsoids and elliptical discs. Proc. Roy. Soc. London A, 2227–2250.

Routh, E.J., 1892. A Treatise On Analytical Statics. vol. 2, Cambridge University Press.

Wang, J., Michelitsch, T.M., Gao, H., 2003. Dynamic fiber inclusions with elliptical and arbitrary cross-sections in a quasiplane piezoelectric medium. Int. J. Solids Struct. 40, 6307–6333.